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Time-dependent weak values and their intrinsic phases of evolution

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Abstract

The equation of motion for a time-dependent weak value of a quantum-mechanical observable is known to contain a complex valued energy factor (the weak energy of evolution) that is defined by the dynamics of the pre-selected and post-selected states which specify the observable's weak value. In this paper, the mechanism responsible for the creation of this energy is identified and it is shown that the cumulative effect over time of this energy is manifested as dynamical phases and pure geometric phases (the intrinsic phases of evolution) which govern the evolution of the weak value during its measurement process. These phases are simply related to a Pancharatnam phase and Fubini–Study metric distance defined by the Hilbert space evolution of the associated pre-selected and post-selected states. A characterization of time-dependent weak value evolution as Pancharatnam phase angle rotations and Fubini–Study distance scalings of a vector in the Argand plane is discussed as an application of this relationship. The theory of weak values is also reviewed and simple 'gedanken experiments' are used to illustrate both the time-independent and the time-dependent versions of the theory. It is noted that the direct experimental observation of the weak energy of evolution would strongly support the time-symmetric paradigm of quantum mechanics and it is suggested that weak value equations of motion represent a new category of nonlocal equations of motion.

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1. Introduction

The theoretical notion of the weak value of a quantum-mechanical observable was introduced by Aharonov *et al* [1–3] over two decades ago. This quantity is the statistical result of a standard measurement procedure performed upon a pre-selected and post-selected (PPS) ensemble of quantum systems when the interaction between the measurement apparatus and each system is sufficiently weak. Unlike the standard strong measurement of a quantum-mechanical observable which significantly disturbs the measured system (i.e., 'collapses its wavefunction'), a weak measurement of an observable for a PPS system does not appreciably disturb the quantum system and yields the weak value as the measured value of the observable. Weak values reflect the nature of a virtually undisturbed quantum reality that exists between the boundaries defined by the PPS states.

A series of experiments performed in recent years has verified aspects of weak value theory [4–8] and the theory has been applied in such diverse areas as contextuality [9], quantum stochastic processes [10, 11], quantum trajectory theory [12], tunnelling and arrival times [13–16], non-locality and consistent histories [17–19], Hardy's paradox [20], superluminality and negative kinetic energy [21, 22], momentum transfer in *welcher Weg* experiments [23], nonclassicality of coherent states and thermal radiation [24, 25], semiclassical weak value theory [26], quantum measurement theory [27–35] and quantum communications [36, 37].

The equation of motion for a time-dependent weak value was first introduced by Parks *et al* [5] who noted that a peculiar energy—the *weak energy of evolution*—naturally appears as a factor in this equation. This energy occurs at the time of interaction between the measurement apparatus and the quantum system during the measurement of a weak value of a system observable when the associated PPS states are explicitly time dependent. Although this energy is not directly measured during this process, it nonetheless has the mathematical form of the weak value of the difference between the two Hamiltonian operators $\widehat{H}_i$ and $\widehat{H}_f$ which describe the evolution of the PPS states, respectively. Since its discovery certain geometric and dynamical properties of this energy have been studied [38] and its significance has been discussed within the context of Hamilton's Principle [39].

The purposes of this paper are to identify the mechanism through which the weak energy of evolution is created at interaction time and to examine how this energy determines the value and influences the evolution of a time-dependent weak value during the measurement process. The new results found and reported here are (1) while the actions of $\widehat{H}_i$ and $\widehat{H}_f$ upon their associated PPS states take place at pre-selection and post-selection times earlier and later than the interaction time t, respectively, the creation of the weak energy of evolution at t is a consequence of the forward and backward time evolutions of these actions to t; (2) the accumulation of these actions over time are physically manifested as dynamical phases and complex valued pure geometric phases—the *intrinsic phases of evolution*—which determine the weak value of the observable at t; (3) these intrinsic phases are related to a Pancharatnam phase angle and a Fubini–Study metric distance associated with the Hilbert space evolutions of the PPS states; and (4) as an application, this relationship provides a simple Argand plane vector representation of weak value evolution in terms of associated Pancharatnam phase angle rotations and Fubini–Study metric distance scalings.

The remainder of this paper is organized as follows: in the next section weak measurement/weak value theory is reviewed and illustrated using a simple photon polarization measurement 'gedanken experiment'. The equation of motion for a time-dependent weak value is derived in section 3 and the associated forward-backward time evolution mechanism that creates the weak energy of evolution at interaction time is formally identified. The intrinsic phases of evolution are defined in section 4. Section 5 establishes both the geometric nature of these phases and their relationship to a Pancharatnam phase angle and a Fubini–Study distance defined by the PPS states. This relationship is applied in section 6 to describe the evolution of a time-dependent weak value in the complex plane in terms of vector rotations and length scalings. In section 7, a time-dependent generalization of the section 2 'gedanken experiment' is used to illustrate the theory developed in the preceding sections. Concluding remarks comprise the final section of this paper.

2. Weak measurements and weak values

2.1. Theory

Consider the von Neumann description of a standard quantum measurement at time t_0 of a time-independent observable $\widehat{A}$ which describes a quantum system in the initial fixed state given by $|\psi_i\rangle = \sum_k c_k |a_k\rangle$, with $\widehat{A}|a_k\rangle = a_k|a_k\rangle$. The Hamiltonian for the interaction between the quantum system and the measuring device is

$$\widehat{H} = \gamma(t)\widehat{A}\widehat{p}, \tag{1}$$

where $\gamma(t) = \gamma\delta(t - t_0)$ defines the strength of an impulsive interaction at time t_0 (note that the time of measurement is defined to be the time of interaction) and $\widehat{p}$ is the momentum operator for the measuring device's pointer that is conjugate to the pointer's position operator $\widehat{q}$. Let $|\phi\rangle$ be the initial state of the pointer of the measurement apparatus and assume that $\langle q|\phi\rangle \equiv \phi(q)$ is real valued with $\langle q\rangle \equiv \langle\phi|\widehat{q}|\phi\rangle = 0$.

Before the interaction occurs, the initial state of the combined pre-selected system and measurement pointer is the tensor product state $|\psi_i\rangle|\phi\rangle$. Immediately after the measurement's impulsive interaction the combined system is in the state

$$|\Phi\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\gamma\widehat{A}\widehat{p}}|\psi_i\rangle|\phi\rangle = \sum_k c_k\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\gamma a_k\widehat{p}}|a_k\rangle|\phi\rangle, \tag{2}$$

where use has been made of the fact that $\int \widehat{H}\,\mathrm{d}t = \gamma\widehat{A}\widehat{p}$. The exponential factor in this equation is the translation operator $\widehat{S}(\gamma a_k)$ for $|\phi\rangle$ in its q-representation. The action of $\widehat{S}(\gamma a_k)$ upon $|\phi\rangle$ in the q-representation is $\langle q|\widehat{S}(\gamma a_k)|\phi\rangle = \langle q - \gamma a_k|\phi\rangle \equiv \phi(q - \gamma a_k)$, i.e. it translates the pointer's wavefunction over a distance γa_k parallel to the q-axis. Application of the closure relation $\widehat{1} = \int |q\rangle\,\mathrm{d}q\,\langle q|$ to equation (2) yields the following expression for $|\Phi\rangle$ in terms of the q-representation of the measurement pointer:

$$|\Phi\rangle = \sum_k c_k \int \langle q|\widehat{S}(\gamma a_k)|\phi\rangle|a_k\rangle|q\rangle\,\mathrm{d}q.$$

The q-representation of this state is

$$\langle q|\Phi\rangle = \sum_k c_k \langle q|\widehat{S}(\gamma a_k)|\phi\rangle|a_k\rangle,$$

where use has been made of the fact that $\langle q|q'\rangle = \delta(q - q')$ and $\int f(q')\delta(q - q')\,\mathrm{d}q' = f(q)$.

When the measurement interaction is strong, the quantum system is appreciably disturbed and its state collapses to an eigenstate $|a_n\rangle$ leaving the pointer in the state $\langle q|\widehat{S}(\gamma a_n)|\phi\rangle$ with probability $|c_n|^2$. Such measurements of an ensemble of identically prepared systems yield the centroid value $\gamma\langle A\rangle \equiv \gamma\langle\psi_i|\widehat{A}|\psi_i\rangle$ of the pointer probability distribution

$$|\langle q|\Phi\rangle|^2 = \sum_k |c_k|^2 |\langle q|\widehat{S}(\gamma a_k)|\phi\rangle|^2 \tag{3}$$

as the measured value of $\widehat{A}$. Note that if the pointer position uncertainty Δq is sufficiently small, then equation (3) is comprised of separated narrow peaks each centred on an eigenvalue a_k.

In contrast to strong measurements, a weak measurement of $\widehat{A}$ occurs when the interaction strength γ is small so that the system is essentially undisturbed and Δq is much larger than the eigenvalue separation. In this case, equation (3) is the superposition of broad, strongly overlapping $|\langle q|\widehat{S}(\gamma a_k)|\phi\rangle|^2$. Even though a single measurement provides little information about $\widehat{A}$, many repetitions allow equation (3) and its associated centroid to be determined to any desired accuracy.

If—after a weak measurement—the fixed system state $|\psi_f\rangle = \sum_k c'_k|a_k\rangle$ is post-selected such that $\langle\psi_f|\,\psi_i\rangle \neq 0$, then the resulting pointer state $|\Psi\rangle$ is given by

$$|\Psi\rangle \equiv \langle\psi_f|\Phi\rangle = \sum_k c'^*_k c_k \int \langle q|\widehat{S}(\gamma a_k)|\phi\rangle|q\rangle\,\mathrm{d}q. \tag{4}$$

Since

$$\widehat{S}(\gamma a_k) = \sum_{m=0}^{\infty} \frac{[-\mathrm{i}\gamma a_k \widehat{p}/\hbar]^m}{m!},$$

then

$$\sum_k c'^*_k c_k \widehat{S}(\gamma a_k) = \sum_k c'^*_k c_k \left\{1 - \frac{\mathrm{i}}{\hbar}\gamma\widehat{p}A_w + \sum_{k=2}^{\infty}\frac{[-\mathrm{i}\gamma\widehat{p}/\hbar]^k}{k!}(A^k)_w\right\},$$

where

$$(A^n)_w = \frac{\sum_k c'^*_k c_k a_k^n}{\sum_k c'^*_k c_k} = \frac{\langle\psi_f|\widehat{A}^n|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle},$$

with the weak value A_w of $\widehat{A}$ given by

$$A_w \equiv (A^1)_w = \frac{\langle\psi_f|\widehat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \tag{5}$$

It is clear from this that A_w is—in general—a complex valued quantity which can be directly calculated from the associated theory. Also note that when $|\psi_i\rangle$ and $|\psi_f\rangle$ are nearly orthogonal the real part $\mathrm{Re}\,A_w$ of A_w can lie far outside the spectrum of eigenvalues for $\widehat{A}$, i.e., its values are *eccentric weak values*.

When the approximation

$$\sum_k c'^*_k c_k \widehat{S}(\gamma a_k) \approx \left\{\sum_k c'^*_k c_k\right\}\widehat{S}(\gamma A_w) \tag{6}$$

is valid, then equation (4) becomes

$$|\Psi\rangle \approx \left\{\sum_k c'^*_k c_k\right\}\int \langle q|\widehat{S}(\gamma A_w)|\phi\rangle|q\rangle\,\mathrm{d}q$$

so that

$$|\langle q|\Psi\rangle|^2 \approx \left|\sum_k c'^*_k c_k\right|^2 |\langle q|\widehat{S}(\gamma\mathrm{Re}\,A_w)|\phi\rangle|^2. \tag{7}$$

This corresponds to a broad pointer position distribution with a single peak at $\langle q\rangle = \gamma\mathrm{Re}\,A_w$ and occurs when the following conditions on the pointer momentum uncertainty Δp are satisfied [40]:

$$\Delta p \ll \frac{\hbar}{\gamma}|A_w|^{-1} \qquad \text{and} \qquad \Delta p \ll \min_{(n=2,3,\ldots)} \frac{\hbar}{\gamma}\left|\frac{A_w}{(A^n)_w}\right|^{\frac{1}{n-1}}. \tag{8}$$

In expressing equation (7), use has been made of the fact that since $\langle q|\phi\rangle$ is real valued, the pointer position must be translated by $\gamma\mathrm{Re}\,A_w$. The imaginary part $\mathrm{Im}\,A_w$ of A_w influences the mean of the pointer's momentum and translates it from the initial mean by an amount proportional to the product of $\mathrm{Im}\,A_w$ with the variance of the initial pointer momentum distribution [34].

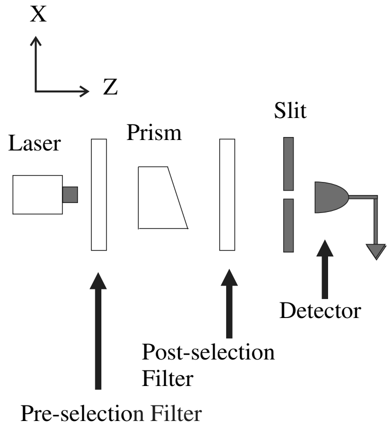

Figure 1. An apparatus for measuring the weak value of the photon linear polarization operator.

2.2. An example

In order to illustrate the theory of weak measurements and weak values, consider a 'gedanken experiment' in which the apparatus depicted in figure 1 measures the weak value of the photon linear polarization operator $\widehat{Q}$. Here linearly polarized photons are created by passing an unpolarized laser beam propagating along the positive z-axis of the laboratory Cartesian reference frame through a pre-selection polarization filter. A birefringent prism segregates these photons into two beams according to polarization state. Their passage through a second polarization filter and then a narrow slit provides the necessary post-selection measurement and the spatial distribution required for observation by a detector.

The x- and y-directions in the laboratory frame define the vertical and horizontal polarization eigenstates $|+\rangle$ and $|-\rangle$, respectively, with $\widehat{Q}|\pm\rangle = \pm|\pm\rangle$, $\langle\pm|\pm\rangle = 1$ and $\langle\pm|\mp\rangle = 0$. Each filter's transmission axis lies in the x–y plane with fixed transmission axis angle settings referenced to the x-direction. The birefringent prism is specially cut and oriented in the laboratory frame to provide a very slight x-component of momentum to the photons passing through it according to the interaction Hamiltonian (equation (1))

$$\widehat{H} = \gamma\delta(t - t_0)\widehat{Q}\widehat{p}_x.$$

Here the pointer state of the apparatus is represented by the Gaussian distribution of beam photons in the x-direction with $\langle x\rangle = 0$ as its initial peak value.

Let the transmission angle settings for the pre- and post-selection polarization filters be fixed at α and β, respectively, so that the associated PPS states are $|\psi_i\rangle = \cos\alpha|+\rangle + \sin\alpha|-\rangle$ and $|\psi_f\rangle = \cos\beta|+\rangle + \sin\beta|-\rangle$. After the post-selection measurement, the pointer state is (equation (4))

$$|\Psi\rangle = \cos\alpha\cos\beta\int\langle x|\,\widehat{S}(+\gamma)|\phi\rangle|x\rangle\,\mathrm{d}x + \sin\alpha\sin\beta\int\langle x|\,\widehat{S}(-\gamma)|\phi\rangle|x\rangle\,\mathrm{d}x.$$

When equation (6) holds, then the last equation provides the following observed translated pointer state distribution in the x-direction with a single peak at $\langle x\rangle = \gamma\mathrm{Re}\,Q_w = \gamma Q_w$ (equation (7)):

$$|\langle x|\Psi\rangle|^2 \approx \cos^2(\alpha - \beta)|\langle x|\widehat{S}(\gamma Q_w)|\phi\rangle|^2, \tag{9}$$

where

$$Q_w = \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)}$$

has been computed directly from equation (5). It is easy to see from this that the values for Q_w can be eccentric, e.g., $Q_w > 1$ when $\alpha = \frac{\pi}{4}$ and $-\frac{\pi}{4} < \beta < 0$ (compare this with $\langle Q\rangle \equiv \langle\psi_i|\,\widehat{Q}|\psi_i\rangle = \cos 2\alpha \in [-1, 1]$).

Approximation (9) is valid when the inequalities (conditions (8))

$$\Delta p_x \ll \frac{\hbar}{\gamma}|Q_w|^{-1} \qquad \text{and} \qquad \Delta p_x \ll \min_{(n=2,3,\ldots)} \frac{\hbar}{\gamma}\left|\frac{Q_w}{(Q^n)_w}\right|^{\frac{1}{n-1}}$$

are simultaneously satisfied. These conditions can be simplified and related to the pointer distribution width by noting that (i) for even n, $\widehat{Q}^n = \widehat{1}$ so that $(Q^n)_w = 1$ and for odd n, $\widehat{Q}^n = \widehat{Q}$ so that $(Q^n)_w = Q_w$; and (ii) if Δx is characterized by the Gaussian width δx of the pointer distribution, then $\Delta p_x \sim \frac{\hbar}{\delta x}$. Straightforward application of (i) and (ii) yields

$$\text{(a)}\quad \delta x \gg \gamma|Q_w| \qquad \text{and} \qquad \text{(b)}\quad \delta x \gg \left(\frac{\gamma}{\min_{(n=2,4,\ldots)}\left\{1, |Q_w|^{\frac{1}{n-1}}\right\}}\right). \tag{10}$$

It is interesting to see how these conditions are used to ensure the validity of approximation (9) in the 'gedanken experiment' when a $|Q_w| > 1$ measurement is to be made. In this case condition (b) becomes $\delta x \gg \gamma$ and (a) is the dominant condition which must be satisfied since satisfaction of (a) also satisfies (b). Restating condition (a) as

$$\frac{\gamma|Q_w|}{\delta x} \ll 1$$

makes it clear that measurements of eccentric values of Q_w require pointer state distribution widths that are much greater than its associated *x*-direction peak translations.

3. Creation mechanism for the weak energy of evolution

Fundamental to the theory of weak values is the proposition that although the measurement of $\widehat{A}$ occurs at time t_0, the PPS states appearing in equation (5) are actually pre-selected and post-selected at times $t_i < t_0$ and $t_f > t_0$, respectively. PPS states selected at these times define past and future boundary conditions which influence A_w at measurement time t_0 via their unitary evolutions forward in time from t_i to t_0 and backward in time from t_f to t_0.

Such unitary evolutions are also responsible for the creation of the weak energy of evolution at the time of measurement. In order to understand this mechanism, consider the time ordered set $\mathcal{A}(t) \equiv \{A_w(t) : t \in T\}$, where $A_w(t)$ is the theoretical weak value of $\widehat{A}$ at interaction time t and $T \equiv [t_1, t_2]$ is a fixed closed time interval such that (i) at each time $t \in T$ the weak value $A_w(t)$ is defined by a state $|\psi_i(t_i)\rangle$ which has been pre-selected at time $t_i = t - \Delta t_i$ and by a state $|\psi_f(t_f)\rangle$ which will be post-selected at time $t_f = t + \Delta t_f$, where Δt_i and Δt_f are fixed time intervals; and (ii) these PPS states continuously change from their initial states at times $t_i = t_1 - \Delta t_i$ and $t_f = t_1 + \Delta t_f$, respectively, in accordance with the Schrödinger equations

$$\frac{\mathrm{d}|\psi_i(t_i)\rangle}{\mathrm{d}t_i} = -\frac{\mathrm{i}}{\hbar}\widehat{H}_i|\psi_i(t_i)\rangle, \tag{11}$$

$t_i \in [t_1 - \Delta t_i, t_2 - \Delta t_i]$, and

$$\frac{\mathrm{d}|\psi_f(t_f)\rangle}{\mathrm{d}t_f} = -\frac{\mathrm{i}}{\hbar}\widehat{H}_f|\psi_f(t_f)\rangle, \tag{12}$$

$t_f \in [t_1 + \Delta t_f, t_2 + \Delta t_f]$. Clearly, such PPS state dynamics can occur naturally or—as will be the case in the 'gedanken experiment' discussed below in section 7—they can be artificially induced. For future reference, the set $\mathcal{A}(t)$ is called the theoretical evolutionary profile for $A_w(t)$ over T, the set $\mathcal{A}_q(t) \equiv \{\mathrm{Re}\, A_w(t) : t \in T\}$ is the theoretical pointer position profile of weak value observations, and the set $\mathcal{A}_p(t) \equiv \{\mathrm{Im}\, A_w(t) : t \in T\}$ is the theoretical profile of values which effect the pointer momentum distribution.

The discussion of weak values in the last section tacitly assumes that the PPS states used there are fixed and do not change with time—so that effectively $\widehat{H}_i = \widehat{0} = \widehat{H}_f$. However, when the Hamiltonians $\widehat{H}_i$ and $\widehat{H}_f$ are non-vanishing and explicitly time independent, then the unitary evolutions of these PPS states to the measurement time t are given by

$$|\psi_i(t)\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\widehat{H}_i \Delta t_i}|\psi_i(t_i)\rangle \equiv \widehat{U}|\psi_i(t_i)\rangle \quad \text{and} \quad |\psi_f(t)\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widehat{H}_f \Delta t_f}|\psi_f(t_f)\rangle \equiv \widehat{V}|\psi_f(t_f)\rangle. \tag{13}$$

Application of these results to equation (5) yields the following time explicit form for the weak value of $\widehat{A}$:

$$A_w(t) = \frac{\langle\psi_f(t_f)|\widehat{V}^\dagger \widehat{A}\widehat{U}|\psi_i(t_i)\rangle}{\langle\psi_f(t_f)|\widehat{V}^\dagger \widehat{U}|\psi_i(t_i)\rangle} = \frac{\langle\psi_f(t)|\widehat{A}|\psi_i(t)\rangle}{\langle\psi_f(t)|\psi_i(t)\rangle}. \tag{14}$$

When these conditions prevail, equations (11) and (12) apply when t_i and t_f are replaced throughout by $t \in T$. This is the case because $[\widehat{U}, \widehat{H}_i] = 0 = [\widehat{V}, \widehat{H}_f]$ so that the actions of $\widehat{H}_i$ and $\widehat{H}_f$ upon the associated PPS states at times t_i and t_f are transformed by the operators $\widehat{U}$ and $\widehat{V}$ into actions of these Hamiltonian operators upon the evolved PPS states at the measurement time t. In particular, from equations (13)

$$\frac{\mathrm{d}|\psi_i(t)\rangle}{\mathrm{d}t} = \widehat{U}\frac{\mathrm{d}|\psi_i(t_i)\rangle}{\mathrm{d}t_i}\frac{\mathrm{d}t_i}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\widehat{U}\widehat{H}_i|\psi_i(t_i)\rangle = -\frac{\mathrm{i}}{\hbar}\widehat{H}_i|\psi_i(t)\rangle \tag{15}$$

and

$$\frac{\mathrm{d}|\psi_f(t)\rangle}{\mathrm{d}t} = \widehat{V}\frac{\mathrm{d}|\psi_f(t_f)\rangle}{\mathrm{d}t_f}\frac{\mathrm{d}t_f}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\widehat{V}\widehat{H}_f|\psi_f(t_f)\rangle = -\frac{\mathrm{i}}{\hbar}\widehat{H}_f|\psi_f(t)\rangle, \tag{16}$$

where use is made of the fact that $\widehat{U}$ and $\widehat{V}$ are constant operators.

If $\dot{A}_w(t) \equiv \frac{\mathrm{d}A_w(t)}{\mathrm{d}t}$ exists at each $t \in T$, then $A_w(t)$ is a continuous function over T and the associated equation of motion for $A_w(t)$ can be obtained by taking the time derivative of equation (14). This initially yields

$$\dot{A}_w(t) = \left\{\frac{\mathrm{d}\langle\psi_f(t)|}{\mathrm{d}t}\widehat{A}|\psi_i(t)\rangle + \langle\psi_f(t)|\frac{\mathrm{d}\widehat{A}}{\mathrm{d}t}|\psi_i(t)\rangle + \langle\psi_f(t)|\widehat{A}\frac{\mathrm{d}|\psi_i(t)\rangle}{\mathrm{d}t}\right\}\langle\psi_f(t)|\psi_i(t)\rangle^{-1}$$
$$- \langle\psi_f(t)|\widehat{A}|\psi_i(t)\rangle\left\{\frac{\mathrm{d}\langle\psi_f(t)|}{\mathrm{d}t}|\psi_i(t)\rangle + \langle\psi_f(t)|\frac{\mathrm{d}|\psi_i(t)\rangle}{\mathrm{d}t}\right\}\langle\psi_f(t)|\psi_i(t)\rangle^{-2}.$$

After applying equations (15) and (16), requiring that operator $\widehat{A}$ be time independent, and rearranging the result, it is found that the equation of motion for $A_w(t)$ is given by

$$\dot{A}_w = \frac{\mathrm{i}}{\hbar}\{(H_f A - A H_i)_w - A_w(H_f - H_i)_w\}. \tag{17}$$

The explicit time dependence of the quantities in this expression has been suppressed for the sake of notational simplicity. However, each w subscripted quantity appearing in this equation depends upon the measurement time t because it is defined in terms of t-dependent PPS states.

The peculiar factor $(H_f - H_i)_w$ appearing in the second term of equation (17) is *the weak energy of evolution* for the PPS system. It is clearly contemporaneous with the measurement time t and since it has the mathematical form of the weak value of $\widehat{H}_f - \widehat{H}_i$, it can be determined theoretically when $\widehat{H}_f - \widehat{H}_i$ and the PPS states are known. Also, observe that if $|\psi_i(t)\rangle = |\psi_f(t)\rangle$ with $\widehat{H}_f = \widehat{H}_i = \widehat{H}$, then the weak energy of evolution vanishes and equation (17) assumes the form of the usual equation of motion for the mean value of $\widehat{A}$ given by

$$\dot{A}_w \longrightarrow \langle \dot{A} \rangle = \frac{\mathrm{i}}{\hbar}\langle [\widehat{H}, \widehat{A}] \rangle. \tag{18}$$

Several additional relationships between $A_w(t)$ and the weak energy of evolution can be deduced from equation (17). First, suppose that $|\psi_i(t)\rangle$ and $|\psi_f(t)\rangle$ are non-orthogonal stationary states of $\widehat{H}_i$ and $\widehat{H}_f$, respectively, with $\widehat{H}_i \neq \widehat{H}_f$. In this case—since $\widehat{H}_i|\psi_i(t)\rangle = E_i|\psi_i(t)\rangle$ and $\widehat{H}_f|\psi_f(t)\rangle = E_f|\psi_f(t)\rangle$—then $(H_f A - AH_i)_w = (E_f - E_i)A_w$ and $(H_f - H_i)_w = (E_f - E_i)$ so that $\dot{A}_w = 0$ and A_w is constant with time. Also, inspection of equation (17) reveals the following special case results: (a) if $\mathcal{A}_p(t) = \varnothing$ (the empty set) so that $\mathcal{A}(t) = \mathcal{A}_q(t)$, then both $(H_f A - AH_i)_w$ and $(H_f - H_i)_w$ are pure imaginary quantities during the time interval T; and (b) if $\mathcal{A}_q(t) = \varnothing$ so that $\mathcal{A}(t) = \mathrm{i}\mathcal{A}_p(t)$, then $(H_f A - AH_i)_w$ is real valued and $(H_f - H_i)_w$ is pure imaginary during T.

The utilization of equations (15) and (16) in the derivation of equation (17) identifies the mechanism responsible for creating the weak energy of evolution at measurement time. The associated identities $\widehat{H}_i|\psi_i(t)\rangle = \widehat{U}\widehat{H}_i|\psi_i(t_i)\rangle$ and $\widehat{H}_f|\psi_f(t)\rangle = \widehat{V}\widehat{H}_f|\psi_f(t_f)\rangle$ formally *establish the forward and backward time evolutions of the Hamiltonian actions upon the PPS states at times $t_i < t$ and $t_f > t$ as the mechanism which creates the weak energy of evolution at measurement time t.* It is important to emphasize—as the above development has shown—that $(H_f - H_i)_w$ is an artefact of dynamics of the PPS states and is not a quantity that is directly measured during the measurement of A_w. However, this artefact is physically significant because it defines phases and associated phase factors that play crucial roles in determining A_w at measurement time when A_w is time dependent. This is the focus of the following section where the form of the general solution to equation (17) is used to identify these phases and phase factors.

4. Intrinsic phases of evolution

Finding the solution to equation (17) requires application of the exponential integrating factor $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t}(H_f - H_i)_w \mathrm{d}t'}$. Using this factor, it is easily determined that the general solution for $A_w(t)$ when $t \in T$ is

$$A_w(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t}(H_f - H_i)_w \mathrm{d}t'} \left\{ A_w(t_1) + \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t'}(H_f - H_i)_w \mathrm{d}t''}(H_f A - AH_i)_w \,\mathrm{d}t' \right\}. \tag{19}$$

Note that this general solution is consistent with the properties of A_w discussed in the last section. For example, if $|\psi_i(t)\rangle = |\psi_f(t)\rangle$ and $\widehat{H}_i = \widehat{H}_f$, then equation (19) becomes equation (18)'s general solution for $\langle A \rangle$. Also, if the PPS states are non-orthogonal stationary states, then equation (19)—upon integration by parts of the second term in curly braces—yields the required result $A_w(t) = A_w(t_1)$.

The form of this solution shows that the time-integrated (i.e., time accumulated) weak energy of evolution is an intrinsic attribute of $A_w(t)$. It is also clear that this quantity determines and influences $A_w(t)$ through the associated phase factors which have been introduced into the general solution by the integrating factor. As will be shown below, the time-integrated weak energy of evolution is equal to the sum of a real valued dynamical phase difference and a complex valued pure geometric phase difference. These dynamical and geometric phases are accumulated by the system as the PPS states evolve during the time interval T.

It is obvious that since $(H_f - H_i)_w$ can be complex valued, then so can the time-integrated weak energy of evolution. In order to determine the physical significance of this phase and analyse the influence of the associated phase factors upon $A_w(t)$, it is useful to relate the time-integrated weak energy to more familiar physical quantities. Since this integral can be written as

$$\frac{1}{\hbar}\int_{t_1}^{t}(H_f - H_i)_w\,\mathrm{d}t' = \frac{1}{\hbar}\int_{t_1}^{t}(H_f)_w\,\mathrm{d}t' - \frac{1}{\hbar}\int_{t_1}^{t}(H_i)_w\,\mathrm{d}t', \tag{20}$$

this relationship can be more easily identified by separately examining each term on the right-hand side of this equation.

Consider the first term and observe that the action of the Hamiltonian operator $\widehat{H}_f$ upon the state $|\psi_f(t)\rangle$ can be uniquely written as [3]

$$\widehat{H}_f|\psi_f(t)\rangle = \langle H_f\rangle|\psi_f(t)\rangle + \Delta H_f\left|\psi_f^{\perp}(t)\right\rangle, \tag{21}$$

where $\langle H_f\rangle = \langle\psi_f(t)|\widehat{H}_f|\psi_f(t)\rangle$ and $\Delta H_f = \sqrt{\langle H_f^2\rangle - \langle H_f\rangle^2}$. The orthogonal companion state $\left|\psi_f^{\perp}(t)\right\rangle$ belongs to the associated Hilbert subspace which is the orthogonal complement of the subspace containing $|\psi_f(t)\rangle$ and satisfies the conditions $\left\langle\psi_f^{\perp}(t)\middle|\psi_f(t)\right\rangle = 0$ and $\Delta H_f = \left\langle\psi_f^{\perp}(t)\middle|\widehat{H}_f|\psi_f(t)\right\rangle$. Equation (21) provides the following definition for $(H_f)_w$ when the dual form of this equation is first used to form the scalar product with the state $|\psi_i(t)\rangle$ and then this product is divided by $\langle\psi_f(t)|\psi_i(t)\rangle \neq 0$:

$$(H_f)_w = \langle H_f\rangle + \Delta H_f\frac{\left\langle\psi_f^{\perp}(t)\middle|\psi_i(t)\right\rangle}{\langle\psi_f(t)|\psi_i(t)\rangle}.$$

Substituting this expression for the integrand in the first integral on the right-hand side of equation (20) yields

$$\frac{1}{\hbar}\int_{t_1}^{t}(H_f)_w\,\mathrm{d}t' = \delta_f(t) + \beta_f(t), \tag{22}$$

where

$$\delta_f(t) \equiv \frac{1}{\hbar}\int_{t_1}^{t}\langle H_f\rangle\,\mathrm{d}t' \tag{23}$$

is identified as an intrinsic real valued dynamical phase that results from the evolution of the state $|\psi_f(t')\rangle$ during the time interval $[t_1, t]$. The second term in equation (22) defines an intrinsic complex valued phase

$$\beta_f(t) \equiv \frac{1}{\hbar}\int_{t_1}^{t}\Delta H_f\frac{\left\langle\psi_f^{\perp}(t')\middle|\psi_i(t')\right\rangle}{\langle\psi_f(t')|\psi_i(t')\rangle}\,\mathrm{d}t' \tag{24}$$

which results from the evolution of $|\psi_f(t')\rangle$, $\left|\psi_f^{\perp}(t')\right\rangle$, and $|\psi_i(t')\rangle$ during $[t_1, t]$.

In a similar manner, since

$$\widehat{H}_i|\psi_i(t)\rangle = \langle H_i\rangle\,|\psi_i(t)\rangle + \Delta H_i|\psi_i^{\perp}(t)\rangle,$$

then the second integral in equation (20) can be written as the sum

$$\frac{1}{\hbar}\int_{t_1}^{t}(H_i)_w\,\mathrm{d}t' = \delta_i(t) + \beta_i(t), \tag{25}$$

where

$$\delta_i(t) \equiv \frac{1}{\hbar}\int_{t_1}^{t}\langle H_i\rangle\,\mathrm{d}t' \tag{26}$$

is an intrinsic real valued dynamical phase that results from the evolution of the state $|\psi_i(t')\rangle$ during the interval $[t_1, t]$ and

$$\beta_i(t) \equiv \frac{1}{\hbar}\int_{t_1}^{t} \Delta H_i \frac{\langle\psi_f(t')\big|\psi_i^{\perp}(t')\big\rangle}{\langle\psi_f(t')|\psi_i(t')\rangle}\,\mathrm{d}t' \tag{27}$$

is an intrinsic complex valued phase that results from the evolution of $|\psi_i(t')\rangle$, $\big|\psi_i^{\perp}(t')\big\rangle$, and $|\psi_f(t')\rangle$ during $[t_1, t]$.

Equations (22) and (25) can now be used to write equation (20) as

$$\frac{1}{\hbar}\int_{t_1}^{t}(H_f - H_i)_w\,\mathrm{d}t' = \delta_f(t) - \delta_i(t) + \beta_f(t) - \beta_i(t). \tag{28}$$

It can be concluded from this and equation (19) that *the time accumulation of the forward and backward time evolutions of the actions of the Hamiltonians* $\widehat{H}_i$ *and* $\widehat{H}_f$ *upon the associated PPS states to the measurement time t is physically manifested at t as the sum* $\delta_f(t) - \delta_i(t) + \beta_f(t) - \beta_i(t)$ *of the intrinsic phases of evolution which determine and influence* A_w *at t via the associated exponential phase factors*. Although $\delta_f(t)$ and $\delta_i(t)$ have been clearly identified as dynamical phases that result from the evolutions of the post-selected and pre-selected states, respectively, a more complete characterization of the properties of the phases $\beta_f(t)$ and $\beta_i(t)$ is desirable.

5. The pure geometric nature of $\beta_f(t)$ and $\beta_i(t)$

This section shows that the phases $\beta_i(t)$ and $\beta_f(t)$ are purely geometric by demonstrating their invariance under both local $U(1)$ gauge transformations and time reparameterization. In addition, a useful relationship is established between the intrinsic phases of evolution and a Pancharatnam phase angle and a Fubini–Study metric distance defined by the evolving PPS states in the associated Hilbert space.

First consider the local gauge invariance of $\beta_f(t)$. Observe that ΔH_f is invariant under the local $U(1)$ gauge transformation $|\psi_f(t')\rangle \to \mathrm{e}^{\mathrm{i}\theta_f(t')}|\psi_f(t')\rangle$ and that via equation (21) this transformation also implies that $\big|\psi_f^{\perp}(t')\big\rangle \to \mathrm{e}^{\mathrm{i}\theta_f(t')}\big|\psi_f^{\perp}(t')\big\rangle$ when $\Delta H_f \neq 0$. Also, when $|\psi_i(t')\rangle \to \mathrm{e}^{\mathrm{i}\theta_i(t')}|\psi_i(t')\rangle$, then

$$\frac{\big\langle\psi_f^{\perp}(t')\big|\,\mathrm{e}^{-\mathrm{i}\theta_f(t')}\,\mathrm{e}^{\mathrm{i}\theta_i(t')}|\psi_i(t')\rangle}{\langle\psi_f(t')|\,\mathrm{e}^{-\mathrm{i}\theta_f(t')}\,\mathrm{e}^{\mathrm{i}\theta_i(t')}|\psi_i(t')\rangle} = \frac{\big\langle\psi_f^{\perp}(t')\big|\psi_i(t')\rangle}{\langle\psi_f(t')|\psi_i(t')\rangle}$$

and it can be concluded from equation (24) that the phase $\beta_f(t)$ is invariant under local $U(1)$ gauge transformations.

Now reparameterize the time t' as $\tau(t')$ such that $\tau(t')$ is monotone increasing over the interval $[\tau(t_1), \tau(t)]$ and $|\psi_f(t')\rangle = |\psi'_f(\tau(t'))\rangle \equiv |\psi'_f(\tau)\rangle$ with state end points $|\psi'_f(\tau(t_1))\rangle = |\psi_f(t_1)\rangle$ and $|\psi'_f(\tau(t))\rangle = |\psi_f(t)\rangle$. It is easy to see that under this reparameterization $\Delta H_f\,\mathrm{d}t' = \Delta H'_f\,\mathrm{d}\tau$, where $\Delta H'_f = \sqrt{\big\langle H_f'^2\big\rangle - \big\langle H'_f\big\rangle^2}$ with $\widehat{H}'_f|\psi'_f(\tau)\rangle = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}|\psi'_f(\tau)\rangle$. The reparameterization $|\psi_f(t')\rangle = |\psi'_f(\tau(t'))\rangle$ also implies via equation (21) that $\big|\psi_f^{\perp}(t')\big\rangle = \big|\psi_f^{\perp\prime}(\tau(t'))\big\rangle$. Thus,

$$\frac{1}{\hbar}\int_{t_1}^{t}\Delta H_f\frac{\big\langle\psi_f^{\perp}(t')\big|\psi_i(t')\rangle}{\langle\psi_f(t')|\psi_i(t')\rangle}\,\mathrm{d}t' = \frac{1}{\hbar}\int_{\tau(t_1)}^{\tau(t)}\Delta H'_f\frac{\big\langle\psi_f^{\perp\prime}(\tau)\big|\psi'_i(\tau)\rangle}{\langle\psi'_f(\tau)|\psi'_i(\tau)\rangle}\,\mathrm{d}\tau$$

from which it may be concluded that the phase $\beta_f(t)$ is invariant under this time reparameterization.

In a similar manner it can be demonstrated that $\beta_i(t)$ is also $U(1)$ gauge invariant and time reparameterization invariant. Thus, when taken together, these invariance properties show that both *the complex valued intrinsic phases of evolution $\beta_f(t)$ and $\beta_i(t)$ are pure geometric phases* in the sense that their values depend only upon the associated smooth evolutionary paths in projective Hilbert space and remain unchanged for any lifts to smooth monotone-increasing parameterized evolutionary paths in Hilbert space which canonically project onto these paths [41].

The results developed in [39] can be used to define the following Pancharatnam phase angle $\chi(t')$ and Fubini–Study metric distance $s(t')$ associated with the evolutions of the PPS states in Hilbert space:

$$\chi(t') \equiv \arg\langle\psi_f(t')|\psi_i(t')\rangle$$

and

$$s(t') \equiv 2\sqrt{1 - |\langle\psi_f(t')|\psi_i(t')\rangle|^2},$$

where $t' \in [t_1, t]$. It is also shown in [39] that

$$(H_f - H_i)_w = \hbar\dot{\chi} + \mathrm{i}\hbar\left\{\frac{s}{4 - s^2}\right\}\dot{s}.$$

Consequently,

$$\frac{1}{\hbar}\int_{t_1}^{t} \mathrm{Re}(H_f - H_i)_w\,\mathrm{d}t' = \int_{t_1}^{t} \dot{\chi}\,\mathrm{d}t' = \chi(t) - \chi(t_1)$$

and

$$\frac{1}{\hbar}\int_{t_1}^{t} \mathrm{Im}(H_f - H_i)_w\,\mathrm{d}t' = \int_{t_1}^{t} \left\{\frac{s}{4 - s^2}\right\}\dot{s}\,\mathrm{d}t' = \ln\sqrt{\frac{4 - s^2(t_1)}{4 - s^2(t)}}.$$

Using the last two equations in equation (28)—along with the fact that $\beta_f(t)$ and $\beta_i(t)$ are complex valued—yields *the following relationships between the intrinsic phases of evolution and the associated Pancharatnam phase angle and Fubini–Study metric distance*:

$$\delta_f(t) - \delta_i(t) + \mathrm{Re}\,\beta_f(t) - \mathrm{Re}\,\beta_i(t) = \chi(t) - \chi(t_1) \tag{29}$$

and

$$\mathrm{Im}\,\beta_f(t) - \mathrm{Im}\,\beta_i(t) = \ln\sqrt{\frac{4 - s^2(t_1)}{4 - s^2(t)}}. \tag{30}$$

Thus, a phase factor in the general solution for $A_w(t)$ with argument given by the left-hand side of equation (29) is equivalent to a phase factor with the Pancharatnam phase angle difference $\chi(t) - \chi(t_1)$ as its argument. However—because of equation (30)'s logarithmic relationship—a phase factor in the general solution with its argument given by the left-hand side of equation (30) is equivalent to a multiplicative scale factor that is dependent only upon the Fubini–Study metric distance. As an application, these equivalences are used in the next section to provide a simple complex plane vector rotation and length scaling model for the evolution of $A_w(t)$.

6. Application: the evolution $A_w(t)$ in the Argand plane

Equations (29) and (30) can now be used to rewrite equation (19) in the equivalent form

$$A_w(t) = \mathrm{e}^{-\mathrm{i}(\chi(t) - \chi(t_1))}\sqrt{\frac{4 - s^2(t_1)}{4 - s^2(t)}}\left\{\begin{array}{l} A_w(t_1) + \\ \quad \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t} \mathrm{e}^{\mathrm{i}(\chi(t') - \chi(t_1))}\sqrt{\frac{4 - s^2(t')}{4 - s^2(t_1)}}(H_f A - A H_i)_w\,\mathrm{d}t' \end{array}\right\}. \tag{31}$$

However, the influence of the intrinsic phases of evolution (via χ and s) upon $A_w(t)$ is more transparent from an analysis of the phase-induced motion of the vector representation of $A_w(t)$ in the Argand plane. This representation is obtained by defining

$$\vec{A}(t) \equiv \begin{pmatrix} \mathrm{Re}\, A_w(t) \\ \mathrm{Im}\, A_w(t) \end{pmatrix},$$

$$S(t) \equiv \sqrt{\frac{4 - s^2(t_1)}{4 - s^2(t)}},$$

$$\widetilde{R}(t) \equiv \begin{pmatrix} \cos(\chi(t) - \chi(t_1)) & \sin(\chi(t) - \chi(t_1)) \\ -\sin(\chi(t) - \chi(t_1)) & \cos(\chi(t) - \chi(t_1)) \end{pmatrix},$$

and

$$\vec{I}(t) \equiv \frac{1}{\hbar} \int_{t_1}^{t} S^{-1}(t') \widetilde{M}(t') \vec{B}(t')\, \mathrm{d}t',$$

where

$$\widetilde{M}(t') \equiv \begin{pmatrix} \sin(\chi(t') - \chi(t_1)) & \cos(\chi(t') - \chi(t_1)) \\ -\cos(\chi(t') - \chi(t_1)) & \sin(\chi(t') - \chi(t_1)) \end{pmatrix},$$

and

$$\vec{B}(t') \equiv \begin{pmatrix} \mathrm{Re}(H_f A - A H_i)_w \\ \mathrm{Im}(H_f A - A H_i)_w \end{pmatrix},$$

and then using these definitions to rewrite equation (31) as

$$\vec{A}(t) = S(t) \widetilde{R}(t) [\vec{A}(t_1) - \vec{I}(t)]. \tag{32}$$

It is apparent from this equation that the gross features of the motion of $\vec{A}(t)$ in the Argand plane are directly influenced by the intrinsic phases of evolution via the continuous $\widetilde{R}(t)$ rotation of $[\vec{A}(t_1) - \vec{I}(t)]$ about the origin through the Pancharatnam phase angle $\chi(t) - \chi(t_1)$ $(= \delta_f(t) - \delta_i(t) + \mathrm{Re}\, \beta_f(t) - \mathrm{Re}\, \beta_i(t))$ and the associated continuous multiplicative scaling of this rotated vector difference by $S(t)$ $(= \exp(\mathrm{Im}\, \beta_f(t) - \mathrm{Im}\, \beta_i(t)))$. The continuous change in $[\vec{A}(t_1) - \vec{I}(t)]$ is induced by $\vec{I}(t)$ and results from a time-integrated continuously scaled (by $S^{-1}(t')$) action of a Pancharatnam phase angle transformation (i.e., $\widetilde{M}(t')$) upon $\vec{B}(t')$. Therefore, it too depends upon the time-integrated influence of the intrinsic phases of evolution.

It is interesting to consider within the context of these results special cases (a) and (b) that are given in section 3. In both of these cases $(H_f - H_i)_w$ is pure imaginary so that $A_w(t)$ is influenced only $\mathrm{Im}\, \beta_f(t) - \mathrm{Im}\, \beta_i(t)$. Thus, only scaling occurs and there is no continuous rotation of $\vec{A}(t)$ by $\widetilde{R}(t)$ in the Argand plane. For case (a) the quantity $(H_f A - A H_i)_w$ is pure imaginary and equation (32) becomes

$$\mathrm{Re}\, A_w(t) = S(t) \left[\mathrm{Re}\, A_w(t_1) - \frac{1}{\hbar} \int_{t_1}^{t} S^{-1}(t') \mathrm{Im}(H_f A - A H_i)_w\, \mathrm{d}t' \right] \tag{33}$$

and $\mathrm{Im}\, A_w(t) = 0$. For case (b) the quantity $(H_f A - A H_i)_w$ is real and equation (32) becomes

$$\mathrm{Im}\, A_w(t) = S(t) \left[\mathrm{Im}\, A_w(t_1) + \frac{1}{\hbar} \int_{t_1}^{t} S^{-1}(t') \mathrm{Re}(H_f A - A H_i)_w\, \mathrm{d}t' \right]$$

and $\mathrm{Re}\, A_w(t) = 0$.

7. Example

7.1. An illustration of time-dependent weak value theory

In this section a theoretical description of a time-dependent generalization of the 'gedanken experiment' of section 2 is used to illustrate aspects of the theory developed above. A hypothetical 'gedanken experiment' implementation of this time-dependent generalization is also briefly discussed.

In the 'gedanken experiment' of section 2, let $\beta = \omega t$, where ω is a constant angular rotation rate. Then the PPS states at measurement time $t \in T \equiv [0, t_2]$ are

$$|\psi_i(t)\rangle \equiv |\psi_i\rangle = \cos\alpha|+\rangle + \sin\alpha|-\rangle$$

and

$$|\psi_f(t)\rangle = \cos\omega t|+\rangle + \sin\omega t|-\rangle$$

and the theoretical evolutionary profile is

$$\mathcal{Q}(t) = \{Q_w(t) : t \in T\},$$

where

$$Q_w(t) = \frac{\cos(\alpha + \omega t)}{\cos(\alpha - \omega t)} \tag{34}$$

is directly determined from equation (14). Note that $\mathcal{Q}(t) = \mathcal{Q}_x(t)$ and $\mathcal{Q}_{p_x}(t) = \varnothing$ so that this case conforms to special case (a) identified in section 3 (i.e., $(H_f Q - QH_i)_w$ and $(H_f - H_i)_w$ are pure imaginary for $t \in T$).

The form of the general solution given by equation (19) can be used to illustrate how the intrinsic phases of evolution influence $Q_w(t)$ during the measurement process and how they determine the right-hand side of equation (34). Using equations (11) and (12) it is found that $\widehat{H}_i = \widehat{0}$ and $\widehat{H}_f = \hbar\omega\widehat{\sigma}_y$, where $\widehat{\sigma}_y$ is the Pauli y-component spin operator. Consequently, the weak energy of evolution is pure imaginary (as required) and is given by

$$(H_f - H_i)_w = \hbar\omega(\sigma_y)_w = -\mathrm{i}\hbar\omega\tan(\alpha - \omega t).$$

Since $\widehat{H}_i = \widehat{0}$, then $\delta_i(t) = 0 = \beta_i(t)$ (equations (26) and (27)) and since $(H_f - H_i)_w$ is pure imaginary, then $\delta_f(t) = 0 = \operatorname{Re}\beta_f(t)$ (equations (23) and (24)) so that

$$\frac{1}{\hbar}\int_0^t \operatorname{Im}(H_f - H_i)_w\,\mathrm{d}t' = \frac{1}{\hbar}\int_0^t \operatorname{Im}(H_f)_w\,\mathrm{d}t' = \ln\left[\frac{\cos\alpha}{\cos(\alpha - \omega t)}\right] = \operatorname{Im}\beta_f(t).$$

Thus, the phase factors $\mathrm{e}^{\pm\frac{\mathrm{i}}{\hbar}\int_0^t (H_f - H_i)\,\mathrm{d}t'}$ appearing in the general solution become

$$\mathrm{e}^{\pm\frac{\mathrm{i}}{\hbar}\int_0^t (H_f - H_i)\,\mathrm{d}t'} = \mathrm{e}^{\mp\operatorname{Im}\beta_f(t)} = \left[\frac{\cos\alpha}{\cos(\alpha - \omega t)}\right]^{\mp 1}. \tag{35}$$

The intrinsic phases of evolution determine $Q_w(t)$ through the multiplicative operation of these phase factors. In particular, since $Q_w(0) = 1$ (equation (34)), then the contribution of the first term in equation (19) to (34) is

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t (H_f - H_i)_w\,\mathrm{d}t'} Q_w(0) = \frac{\cos\alpha}{\cos(\alpha - \omega t)}.$$

Also, $\widehat{H}_f\widehat{Q} - \widehat{Q}\widehat{H}_i = \widehat{H}_f\widehat{Q} = \hbar\omega\widehat{\sigma}_y\widehat{Q} = \mathrm{i}\hbar\omega\widehat{\sigma}_x$, where $\widehat{\sigma}_x$ is the Pauli x-component spin operator. The influence of the phase factor $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^{t'} (H_f - H_i)_w\,\mathrm{d}t''}$ upon

$$(H_f Q - QH_i)_w = \mathrm{i}\hbar\omega(\sigma_x)_w = \mathrm{i}\hbar\omega\left[\frac{\sin(\alpha + \omega t)}{\cos(\alpha - \omega t)}\right]$$

in the integrand of the second term of the general solution is the product

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^t (H_f - H_i)_w\,\mathrm{d}t'}(H_f Q - Q H_i)_w = \mathrm{i}\hbar\omega\left[\frac{\sin(\alpha+\omega t)}{\cos\alpha}\right].$$

The total accumulated influence of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^t (H_f - H_i)_w\,\mathrm{d}t'}$ upon $(H_f Q - Q H_i)_w$ over the time interval $[0, t]$ is

$$\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^{t'} (H_f - H_i)_w\,\mathrm{d}t''}(H_f Q - Q H_i)_w\,\mathrm{d}t' = \frac{\cos(\alpha+\omega t)}{\cos\alpha} - 1.$$

As was the case for the first term, the second term in the general solution results from the same multiplicative effect of the phase factor $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t (H_f - H_i)_w\,\mathrm{d}t'}$ upon this integrated influence:

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t (H_f - H_i)_w\,\mathrm{d}t'}\left[\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^{t'} (H_f - H_i)_w\,\mathrm{d}t''}(H_f Q - Q H_i)_w\,\mathrm{d}t'\right] = \frac{\cos(\alpha+\omega t)}{\cos(\alpha-\omega t)} - \frac{\cos\alpha}{\cos(\alpha-\omega t)}.$$

Finally, combining these two terms yields the required expression for $Q_w(t)$ given by equation (34).

As mentioned above, this example corresponds to special case (a) identified in section 3. Thus, as discussed at the end of section 6, the general solution for $Q_w(t)$ also has the form of equation (33) from which it can be seen that only scalings occur during the evolution of $Q_w(t)$. The associated scale factors $S(t)$ and $S^{-1}(t)$ are precisely the phase factors of equation (35). No rotations occur because the Pancharatnam phase angle for this case vanishes, i.e. $\chi(t) - \chi(0) = \delta_f(t) - \delta_i(t) + \mathrm{Re}\,\beta_f(t) - \mathrm{Re}\,\beta_i(t) = 0$, so that $\widetilde{R}(t)$ is the identity matrix and $\widetilde{M}(t) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Consequently, the vector evolution of $Q_w(t)$ in the Argand plane takes place entirely on the real x-axis according to

$$\overrightarrow{Q}(t) = Q_w(t)\overrightarrow{e}_x,$$

where $\overrightarrow{e}_x$ is the unit vector along the positive x-axis.

7.2. *Experimental implementations*

Assume that the only objective of a hypothetical implementation of this theory as a 'gedanken experiment' is to obtain an experimentally determined pointer position profile for $Q_w(t)$. In order that the post-selected state be explicitly time dependent, an apparatus is used that consists of N copies of the apparatus depicted in figure 1—except now a small electric motor rotates the post-selection filter of each copy at a constant angular rate ω with period τ around its laboratory frame z-axis of symmetry. The rotation start and stop times are identical for each copy and each records photon detections during the interval $T = [0, \tau]$. The PPS states for each copy of the apparatus at a measurement time t are as given in the previous subsection and each copy is automatically controlled so that conditions (10) are always satisfied and each detector is both 'ideal' (i.e., it is 100% efficient and detects every post-selected photon only) and 'intelligent' (i.e., it records and stores the x-direction position and measurement time for each detection).

After the measurement process is complete, the detections recorded by the N detectors are combined and partitioned into $M = \frac{\tau}{\delta t}$ (M a counting number) time bins of width δt. N is large enough and δt is small enough so that there are a sufficient number of detections in each bin to provide an 'instantaneous' pointer position distribution profile from which the peak value corresponding to the associated experimentally determined weak value $Q_w^{\mathrm{exp}}(t_\ell)$ is extracted. Here t_ℓ is the measurement time associated with ℓ^{th} bin. If

$\mathcal{Q}_x^{\text{exp}} \equiv \left\{Q_w^{\text{exp}}(t_\ell) : t_\ell \in T, 1 \leqslant \ell \leqslant M\right\}$ is the experimental pointer position evolutionary profile, then—when $t_2 = \tau$ and each experimental value is error free— $\mathcal{Q}_x^{\text{exp}} \subset \mathcal{Q}_x(t)$.

Instead of using N copies of the apparatus to perform N identical experiments in parallel (the first method), a single apparatus can be used to perform M experiments sequentially (the second method). In this case—however—the post-selected state is not explicitly time dependent: the post-selection filter is fixed for each experiment such that for the ℓth experiment the transmission angle setting is $\beta_\ell = \omega t_\ell$ and the detector only need record photon x-positions to provide a pointer position profile and associated peak value $Q_w^{\text{exp}}(\beta_\ell)$. Each of the sequential M experiments can run as long as required to produce its pointer profile. Unlike the first method, the second method does not require combining and partitioning the data. Ideally $Q_w^{\text{exp}}(\beta_\ell) = Q_w^{\text{exp}}(t_\ell)$ so that the experimental pointer position profiles produced by each method are identical.

Although the second 'static' method likely requires more time to perform the measurements, if the use of an explicitly time-dependent post-selected state is not required (as is the case here since the stated objective is to produce a pointer position profile for Q_w), then it is preferable to the first method because it requires much fewer resources and is less complex from a procedural perspective. Obviously, if explicitly time-dependent states are an experimental requirement, then a 'static' approach (exemplified by the second method) cannot be used. This would be the case—for example—if the experimental objective is to somehow directly observe the weak energy of evolution at interaction time, since time-dependent states are required for its creation.

8. Closing remarks

Weak value theory is a special consequence of the time-symmetric reformulation of quantum mechanics (TSQM). Whereas standard quantum mechanics (QM) describes a quantum system at a time t using a state evolving forward in time from the past to t, TSQM also uses a second state evolving backward in time from the future to t. Although TSQM has predicted new experimentally verified effects which seem impossible according to QM, it is a reformulation of QM which is consistent with all the predictions made by QM. It therefore seems unlikely that experiments will be able to directly confirm the forward/backward time evolution of states (FBTE) interpretation of TSQM [42]. However, this paper has identified the forward/backward time evolution of Hamiltonian actions upon states as an additional new consequence of TSQM. As the theory developed above shows, a manifestation of this is the weak energy of evolution which appears at the time of measurement t of a weak value. It is suggested that a direct experimental observation of this weak energy at t during an $A_w(t)$ measurement process would strongly support the FBTE interpretation. Note that since $\operatorname{Re} \dot{A}_w(t)$ depends upon the weak energy of evolution and also defines the slope of $\mathcal{A}_q(t)$ at t, then estimates of the slope made from measured values of $\operatorname{Re} \dot{A}_w$ at various times can be used to provide an indirect observation of the existence and magnitude of the weak energy of evolution.

Both classical and QM equations of motion (e.g., those for the mean values of observables) assert that the past state of a system determines its future state. As shown above, this is generally not the case for weak value equations of motion since they depend upon the dynamics of future post-selected states. Because of this, these equations of motion can be thought of as being quasi-nonlocal in time. Nonlocal Heisenberg representation equations of motion associated with potential effects and modular variables have already been identified [43, 44]. It is suggested that weak value equations of motion represent another category of nonlocal equations of motion.

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